E-contents on Mathematics for BSc H(III), Paper-VI, Complex analysis

Krishnanand Vishwakarma
(Guest Faculty)

Department of Mathematics
Bihar National college Patna (Bihar), India

E-mail: krishnanandv25@gmail.com

E-contents-05

Contact: 9452259427

April 18, 2020

Topic- Construction of analytic function

Milne-Thomson Method to find an analytic function

Let $u(x,y)$ be a given harmonic function such that $f(z) = u(x,y) + iv(x,y)$ be an analytic function. Then

$$\begin{aligned} f'(z) &= u_x(x,y) + iv_x(x,y) \\ f'(z) &= u_x(x,y) - u_y(x,y) \end{aligned}$$

let $\phi_1(x,y) = u_x(x,y)$ and $\phi_2(x,y) = u_y(x,y)$, Then we have

$$f'(z) = \phi_1\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) - i\phi_2\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Now, putting $z = \bar{z}$ in above equation we get

$$\begin{aligned} f'(z) &= \phi_1(z,0) - i\phi_2(z,0) \\ f(z) &= \int [\phi_1(z,0) - i\phi_2(z,0)]dz + c \end{aligned}$$

Continue...

Note: If harmonic function $v(x,y)$ is given, then the analytic function

$$\boxed{f(z)=\int[\phi_1(z,0)+i\phi_2(z,0)]dz+c}$$

where $\phi_1(z,0)=v_y(z,0)$ and $\phi_2(z,0)=v_x(z,0)$.
Example 1. Show that $u(x,y)=\sin x\cosh y+2\cos x\sinh y+x^2-y^2+4xy$ is harmonic. Find an analytic function $f(z)$ in terms of z.
Solution: here $u(x,y)=\sin x\cosh y+2\cos x\sinh y+x^2-y^2+4xy$, then

$$\begin{aligned}
u_x &= \cos x\cosh y-2\sin x\sinh y+2x+4y \\
u_{xx} &= -\sin x\cosh y-2\cos x\sinh y+2 \qquad (1)\\
u_y &= \sin x\sinh y+2\cos x\cosh y-2y+4x \\
u_{yy} &= \sin x\cosh y+2\cos x\sinh y-2 \qquad (2)
\end{aligned}$$

Adding equation (6) and (7) we get

$$u_{xx}+u_{yy}=0$$

hence $u(x,y)$ is harmonic. Now we have to evaluate $f(z)$ for this

Continue...

$$\phi_1(z,0) = u_x(z,0) = \cos z \cosh 0 - 2\sin z \sinh 0 + 2z + 40 = \cos z + 2z$$
$$\phi_2(z,0) = u_y(z,0) = \sin z \sinh 0 + 2\cos z \cosh 0 - 0 + 4z = 2\cos z + 4z$$

Then by Milne-Thomson method we have

$$\begin{aligned} f(z) &= \int [\phi_1(z,0) - i\phi_2(z,0)]dz + c \\ f(z) &= \int [\cos z + 2z - i(2\cos z + 4z)]dz + c \\ f(z) &= \sin z + z^2 + 2i\sin z - 2iz^2 + c \end{aligned}$$

Which is the required analytic function in terms of z.

Example 2. Given $v(x,y) = x^4 - 6x^2y^2 + y^4$ find $f(z) = u(x,y) + iv(x,y)$ such that $f(z)$ is analytic.

Solution: Here $v(x,y) = x^4 - 6x^2y^2 + y^4$, then $v_x = 4x^3 - 12xy^2$, $v_{xx} = 12x^2 - 12y$, $v_y = -12x^2y + 4y^3$, $v_{yy} = -12x^2 + 12y^2$

$\therefore$

$$u_{xx} + u_{yy} = 0$$

Continue...

Now,

$$\begin{aligned} \phi_1(z,0) &= v_y(z,0) = 0 \\ \phi_2(z,0) &= v_x(z,0) = 4z^3 \end{aligned}$$

Then by Milne-Thomson method we have

$$\begin{aligned} f(z) &= \int [\phi_1(z,0) + i\phi_2(z,0)]dz + c \\ f(z) &= \int [0 + i4z^3]dz + c \\ f(z) &= iz^4 + c \end{aligned}$$

Which is the required analytic function in terms of z.
Example 3. If $u+v=(x-y)(x^2+4xy+y^2)$ and $f(z)=u+iv$ find the analytic function $f(z)$ in terms of z.
Solution: Here

$$u+v=(x-y)(x^2+4xy+y^2)$$

Differentiating above equation w.r.t. x and y respectively we have

Continue...

$$u_x + v_x = (x-y)(2x+4y) + (x^2+4xy+y^2) \tag{3}$$
$$u_y + v_y = (x-y)(4x+2y) - (x^2+4xy+y^2) \tag{4}$$

Since $f(z) = u + iv$ is analytic then u and v satisfy the C-R equations as $u_x = v_y$ and $u_y = -v_x$. Using C-R equation in (9) we have

$$-v_x + u_x = (x-y)(4x+2y) - (x^2+4xy+y^2) \tag{5}$$

Now adding equation (8) and (10) we get

$$2u_x = (x-y)(6x+6y) \implies u_x = 3(x^2-y^2) \tag{6}$$

Again subtracting equation (10) from equation (8) we have

$$2v_x = (x-y)(y-x) + 2x^2 + 8xy + 2y^2 \implies v_x = 6xy \tag{7}$$

Then from equation (12) we have $u_y = -6xy$ (from C-R equation). Now $\phi_1(z,0) = u_x(z,0) = 3z^2$ and $\phi_2(z,0) = u_y(z,0) = 0$.
Then by Milne-Thomson method we have

Continue...

$$
\begin{aligned}
f(z) &= \int [\phi_1(z,0) - i\phi_2(z,0)]dz + c \\
f(z) &= \int [3z^2 - i(0)]dz + c \\
f(z) &= z^3 + c
\end{aligned}
$$

Which is the required analytic function in terms of z.

Exercise

(i)- Find the analytic function $f(z) = u + iv$ of which the real part $u = e^x(x\cos y - y\sin y)$.
(ii)- Find the analytic function $f(z) = u + iv$ where the imaginary part $v(x,y) = 3x^2y - y^3$.
(iii)- Find the analytic function $f(z) = u + iv$ in terms of z given that $u - v = e^x(\cos y - \sin y)$.

(try to solve yourself)

Ans: (i) $ze^z + c$ (ii) $z^3 + c$ (iii) $e^z + c$

If you have any query regarding the content please contact me through email or other sources.