E-contents on Mathematics for BSc H(III), Paper-VI, Complex analysis

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Topic- Analytic function and C-R equation.

Definition

A function $f(z)$ is said to be analytic or holomorphic at the point $z = a \in \mathbb{C}$ if it is differentiable in the neighbourhood of a.

Cauchy-Riemann (C-R) equation in Cartesian form

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function in a domain D, then u and v satisfy the Cauchy-Riemann equation $u_x = v_y \& u_y = -v_x$.

Proof. In order to derive C-R equation let us consider a function $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point $z_0 = x_0 + iy_0$, then $\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists along any path, where $h = h_1 + ih_2$.
Now,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

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$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{u(x_0+h_1,y_0+h_2)+iv(x_0+h_1,y_0+h_2)-u(x_0,y_0)-iv(x_0,y_0)}{h_1+ih_2} \\ &= \lim_{h\to 0}\left[\frac{u(x_0+h_1,y_0+h_2)-u(x_0,y_0)}{h_1+ih_2}\right] \\ &\quad +i\lim_{h\to 0}\left[\frac{v(x_0+h_1,y_0+h_2)-v(x_0,y_0)}{h_1+ih_2}\right] \end{aligned}$$

now taking limit along x-axis, then $h \to h_1 + i0$

$$\begin{aligned} f'(z_0) &= \lim_{h_1\to 0}\left[\frac{u(x_0+h_1,y_0)-u(x_0,y_0)}{h_1}\right] + i\lim_{h_1\to 0}\left[\frac{v(x_0+h_1,y_0)-v(x_0,y_0)}{h_1}\right] \\ f'(z_0 &= u_x(x_0,y_0)+iv_x(x_0,y_0) \qquad (1) \end{aligned}$$

similarly, taking limit along imaginary axis, then $h \to 0 + ih_2$, then we have

$$f'(z_0) = \lim_{h_2\to 0}\left[\frac{u(x_0,y_0+h_2)-u(x_0,y_0)}{ih_2}\right] + i\lim_{h_2\to 0}\left[\frac{v(x_0,y_0+h_2)-v(x_0,y_0)}{ih_2}\right]$$

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$$f'(z_0 \quad = \quad -iu_x(x_0,y_0)+v_y(x_0,y_0) \tag{2}$$

Now from equation (1) and (2) we get

$$u_x(x_0,y_0)=v_y(x_0,y_0)\&u_y(x_0,y_0)=-v_x(x_0,y_0)$$

which is the required C-R equation in Cartesian form.

Cauchy-Riemann (C-R) equation in polar coordinates

Let $f(z)=u(r,\theta)+iv(r,\theta)$ be an analytic function at $z=re^{i\theta}$, then $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$.

Proof. In order to derive C-R equation in polar coordinate let us consider $x=r\cos\theta, y=r\sin\theta$
Now,

$$\frac{\partial u}{\partial r} \quad = \quad \frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

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$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \tag{3}$$

also

$$\begin{aligned}
\frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} \\
\frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta) \\
\frac{1}{r}\frac{\partial v}{\partial \theta} &= -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta
\end{aligned}$$

Since, $f(z)$ is analytic then using $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ in above equation we get

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta \tag{4}$$

Now, comparing equation (3) and (4) we get

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$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, similarly we can derive $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$.
These two equation are C-R equation in polar form.

Remarks

• If $f(z) = u(x,y) + iv(x,y)$ is differentiable at any point Z_0, then the C-R equations holds at that point always.
• If $f(z) = u(x,y) + iv(x,y)$ is continuous at z_0 and also suppose that partial derivative of u and v w.r.t. x and y i.e. u_x, u_y, v_x, v_y are exists and continuous at z_0 also C-R equation holds at z_0, then $f(z)$ is differentiable at z_0.

Exercise

Verify Cauchy-Riemann equation for the following function.
1- $f(z) = z^3$
2- $f(z) = Re(z)$
3- $f(z) = e^x(\cos y - \sin y)$ (try to solve yourself)

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