E-contents on Mathematics for BSc H(III), Paper-VI, Complex analysis

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Topic- Limit of a complex valued functions

Definition

A complex valued function $f(z)$ is said to have a limit l as $z \to z_0$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |z - z_0| < \delta \implies |f(z) - l| < \varepsilon$ i.e. we can write $\lim_{z \to z_0} f(z) = l$.

Results

1- When the limit of the function $f(z)$ exists as $z \to z_0$ then limit has unique value.
2- Let $f(z) = u(z) + iv(z)$, where $u(z) = u(x,y)$ & $v(z) = v(x,y)$ are real-valued function defined on any domain D, $\lim_{z \to z_0} f(z) = l_1 + il_2$ iff $\lim_{(x,y) \to (x_0,y_0)} u(x,y) = l_1$ and $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = l_2$.

Problem. The limit of function $f(z) = \frac{\bar{z}}{z}$, $z \neq 0$ exists ?.

Solution: Here $f(z) = \frac{\bar{z}}{z} = \frac{(\bar{z})^2}{|z|^2} = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2}$
Now equating real & imaginary part of the function of $f(z)$, then we have

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$u(x,y) = \frac{x^2-y^2}{x^2+y^2}$ and $v(x,y) = -\frac{2xy}{x^2+y^2}$
now, we shall check whether the $\lim_{z\to 0} f(z)$ exists or not, by examining the limits as $z \to 0$ in many path. for this we choose a path along $y = mx$, where $m \in \mathbb{R}$ be any real number. Then we have
$\lim_{(x,y)\to(0,0)} u(x,y) = \lim_{x\to 0} \frac{x^2-m^2x^2}{x^2+m^2x^2} = \frac{1-m^2}{1+m^2}$
and $\lim_{(x,y)\to(0,0)} v(x,y) = \lim_{x\to 0} -\frac{2xmx}{x^2+m^2x^2} = -\frac{2m}{1+m^2}$, obviously both the limit depends on m and hence for different value of m limit is different. Hence it is clear that $\lim_{z\to 0} f(z)$ does not exist.

Remarks

• If the function $f(z)$ approaches two different values as $z \to z_0$ along the two different path then $\lim_{z\to z_0} f(z)$ does not exist.
• Let $f(z)$ and $g(z)$ be two functions whose limits exist at $z \to z_0$ that is $\lim_{z\to z_0} f(z) = l_1$ and $\lim_{z\to z_0} g(z) = l_2$, then
(i) $\lim_{z\to z_0} [f(z) \pm g(z)] = l_1 \pm l_2$, (ii) $\lim_{z\to z_0} [f(z)g(z)] = l_1 l_2$
(iii) $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{l_1}{l_2}$, $l_2 \neq 0$.

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Exercise

Prove that the following functions $f(z)$ has limits or not as $z \to 0$.

1- $f(z) = \frac{x^2y^2}{(x+y^2)^3}$, $z \neq 0$.

2- $f(z) = \frac{\sqrt{|xy|}}{x+iy}$, $z \neq 0$.

3- $f(z) = \frac{xy}{x^2+y^2}$, $z \neq 0$ where $z = x + iy$.

(try to solve yourself)

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